Computing abelian extensions of quartic fields using complex multiplication

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$\mathrm{Aut}\left(\mathbb{Q}(\sqrt{29})/\mathbb{Q}\right)$ is generated by $a + b\sqrt{29} \mapsto a - b\sqrt{29}$.

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abelian extension L of K: extension of number fields L/K such that

- $|\mathrm{Aut}(L/K)| = [L : K]$
- $\mathrm{Aut}(L/K)$ is abelian

$*$ means Galois extension

Let K be a number field.

Class Field Theory

There exists a set $\{H_K(m) : m \in \mathbb{Z}\}$ such that:
For any finite degree abelian extension L of K,

there exists $f \in \mathbb{Z}_{>0}$ such that $L \subseteq H_K(f)$.

$H_K(m)$ is called the ray class field of K for the modulus m.

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Kronecker-Weber Theorem (1896)

Let $m \in \mathbb{Z}_{>0}$.
Then $H_{\mathbb{Q}}(m) = \mathbb{Q}(\zeta_m) = \mathbb{Q}(\exp(2\pi i\tau) : \tau \in \frac{1}{m}\mathbb{Z})$.

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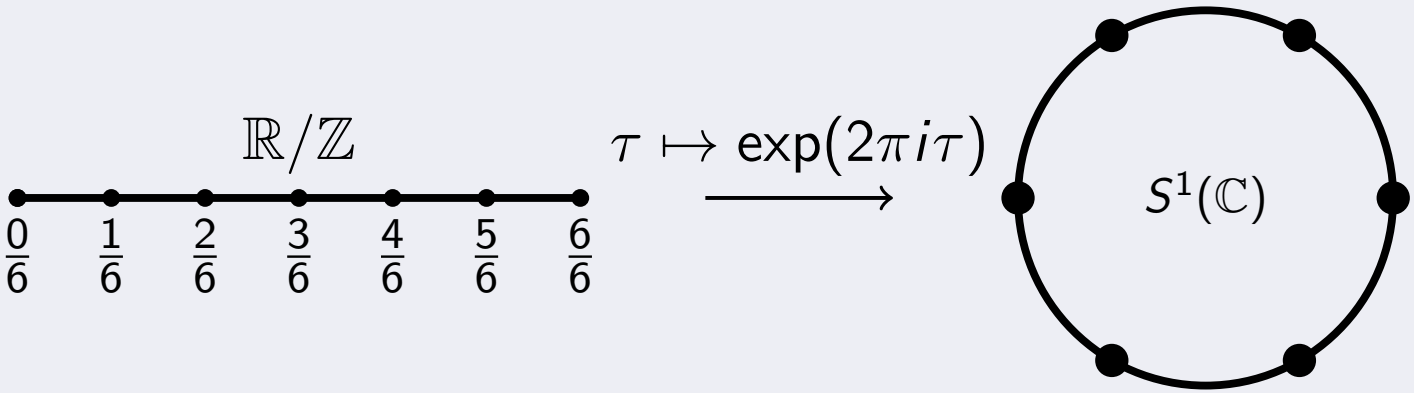

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analytic function: map $\tau \mapsto \exp(2\pi i\tau)$ from $\mathbb{R}/\mathbb{Z}$ to the unit circle
special arguments: torsion points of the group $\mathbb{R}/\mathbb{Z}$

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Hilbert's 12th Problem (1900)

Let K be a number field.
Let $m \in \mathbb{Z}_{>0}$.
Then $H_K(m) =$???.

---

analytic function: ???
special arguments: ???

$K = \mathbb{Q}(\sqrt{-D})$ *with* $D \in \mathbb{Z}_{>0}$

Definition (CM field)

CM field K: number field which is
a totally imaginary quadratic extension (no embeddings of K into $\mathbb{R}$) of
a totally real number field K_0 (all embeddings of K_0 in $\mathbb{C}$ land in $\mathbb{R}$).

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An Oversimplification of CM Theory

Let K be a CM field of degree $2g$.
Let $m \in \mathbb{Z}_{>0}$.
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where τ depends on K and $j(\tau)$ is an invariant of $\mathbb{C}^2/(\mathbb{Z} + \tau\mathbb{Z}) \cong E_\tau$.

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A Miracle

Let K be a CM field of degree 2. Let $m \in \mathbb{Z}_{>0}$. Then

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No miracles here.

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Streng (2010)

Let K be a primitive CM field of degree 4. Let $m \in \mathbb{Z}_{>0}$. Then

$$H_{K_0}(m)\,\mathrm{CM}_K(m) \subseteq H_K(m)$$

is of exponent at most 2.

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Shimura (1968)

Let K be a primitive CM field of degree 4.
Then there is $m \in \mathbb{Z}_{>0}$ such that

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Let $\mathcal{O}_K$ be the ring of integers of K.

$$\mathrm{Cl}_K(1) = \frac{\{ \text{ ideals of } \mathcal{O}_K \}}{\{ \text{ ideals of } \mathcal{O}_K \text{ generated by a single element } \}}$$

A. (2019)

Let K be a primitive CM field of degree 4.
Let S be a finite set of prime ideals of $\mathcal{O}_K$ such that
S contains all prime ideals which divide $2\mathcal{O}_K$ and $|\mathrm{Cl}_K(1)/\langle S \rangle|$ is odd.
Let P be the set of primes of $\mathbb{Z}$ divisible by the elements of S.
Take $m = 4\prod_{p \in P} p$. Then, $(\star)$ is true.

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Let P be the set of rational primes below the prime ideals in S.
Take $m = 4\prod_{p\in P} p$. Then, $H_K(1) \subseteq H_{K_0}(m)\,\mathrm{CM}_K(m)$ is true.

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Let $K = \mathbb{Q}(y)$ where $y^4 + 24y^2 + 28 = 0$.
Let $\mathrm{Cl}_K(1) = \langle[\mathfrak{p}_5]\rangle_8$ where $\mathfrak{p}_5 = 5\mathcal{O}_K + (y-2)\mathcal{O}_K$.

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Then we take $P = \{2, 5\}$.

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Take $m = 40$.

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Then we take $P = \{2, 5\}$.
Take $m = 40$.

$$H_K(1) \subseteq H_{K_0}(40)\,\mathrm{CM}_K(40).$$

$[H_{K_0}(40)\,\mathrm{CM}_K(40) : H_K(1)] = 49\,152.$

Given $m \in \mathbb{Z}_{>0}$.

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By Galois theory, this is equivalent to asking...

Given $m \in \mathbb{Z}_{>0}$.

$$\mathbf{A} \supseteq \mathbf{B} \cap \mathbf{C}\ ?$$

$\mathbf{A} := \mathrm{Aut}(H_K(m)/H_K(1))$

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Computing $\mathrm{Aut}(H_K(m)/\mathrm{CM}_K(m))$.

- $\mathrm{Cl}_K(m)$, $\mathrm{Cl}_K(1)$, $\mathrm{Cl}_{K_0}(m)$ ..... PARI/GP ✓
- N_{K/K_0} ..... PARI/GP ✓
- $\mathfrak{C}_K(m)$, $m = 1$ ..... RECIP (Streng) / CMH (Enge / Thomé) ✓
  $m \geq 1$ ..... (A. 2020) ✓

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$$0 \to \mathcal{O}_{K_0}^+/N_{K/K_0}(\mathcal{O}_{K,1}) \to \mathfrak{C}_K(m) \to \ker(\mathrm{Cl}_K(m) \to \mathrm{Cl}_{K_0}^+(1)) \to 0$$

- operations on group morphisms ..... `abgrp.gp` ✓

  kernels, images, quotients, group extensions

Let $K = \mathbb{Q}(y)$ where $y^4 + 24y^2 + 28 = 0$.

$m = 1$

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$$\begin{aligned}
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\mathbf{B} &= \mathrm{Aut}(H_K(2)/H_{K_0}(2)) &&\cong \langle [\mathfrak{p}_5] \rangle_8 \times && \langle [(\alpha_1)] \rangle_2 \times \langle [(\alpha_2)] \rangle_2 \\
\mathbf{C} &= \mathrm{Aut}(H_K(2)/\mathrm{CM}_K(2)) &&\cong && \langle [(\alpha_1)] \rangle_2
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$\mathbf{A} \supseteq \mathbf{B} \cap \mathbf{C}$? **YES.** $\Rightarrow H_K(1) \subseteq H_{K_0}(2)\,\mathrm{CM}_K(2)$

Let $K = \mathbb{Q}(y)$ where $y^4 + 24y^2 + 28 = 0$.

$m = 1$

$$\begin{aligned}
\mathbf{A} &= \mathrm{Aut}(H_K(1)/H_K(1)) &&\cong \langle [(1)] \rangle_1 \\
\mathbf{B} &= \mathrm{Aut}(H_K(1)/H_{K_0}(1)) &&\cong \langle [\mathfrak{p}_5] \rangle_8 \\
\mathbf{C} &= \mathrm{Aut}(H_K(1)/\mathrm{CM}_K(1)) &&\cong \langle [\mathfrak{p}_5^4] \rangle_2
\end{aligned}$$

$\mathbf{A} \supseteq \mathbf{B} \cap \mathbf{C}$? **NO.** $\Rightarrow H_K(1) \not\subseteq H_{K_0}(1)\,\mathrm{CM}_K(1)$

$m = 2$

$$\begin{aligned}
\mathbf{A} &= \mathrm{Aut}(H_K(2)/H_K(1)) &&\cong && \langle [(\alpha_1)] \rangle_2 \times \langle [(\alpha_2)] \rangle_2 \\
\mathbf{B} &= \mathrm{Aut}(H_K(2)/H_{K_0}(2)) &&\cong \langle [\mathfrak{p}_5] \rangle_8 \times && \langle [(\alpha_1)] \rangle_2 \times \langle [(\alpha_2)] \rangle_2 \\
\mathbf{C} &= \mathrm{Aut}(H_K(2)/\mathrm{CM}_K(2)) &&\cong && \langle [(\alpha_1)] \rangle_2
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$\mathbf{A} \supseteq \mathbf{B} \cap \mathbf{C}$? **YES.** $\Rightarrow H_K(1) \subseteq H_{K_0}(2)\,\mathrm{CM}_K(2)$

$\mathbf{A} \supseteq \mathbf{C}$! $\Rightarrow H_K(1) \subseteq \mathrm{CM}_K(2)$.

Let K be a primitive quartic CM field.

$$H_K(1) \subseteq H_{K_0}(m)\,\mathrm{CM}_K(m) \qquad (\star)$$

So Far...

- Find m such that $\star$ is true. ..... ✓
- Given m, determine if $\star$ is true. ..... ✓

Computing $\mathrm{CM}_K(m)$.

- $m = 1$
- $m = 2$
- $m \neq 1, 2$

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- $m = 2$ ..... example after this slide ✓

  Assume $\star$ for $m = 2$ and $[H_K(1) : K]$ is large:

  can find $H_K(1)$ faster than Kummer theory
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Computing $\mathrm{CM}_K(m)$.

- $m = 1$ ..... RECIP (Streng) / CMH (Enge, Thomé)) ✓ $\sim 80\%$
- $m = 2$ ..... example after this slide ✓ $\sim 10\%$

  Assume $\star$ for $m = 2$ and $[H_K(1) : K]$ is large:

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- $m \neq 1, 2$ ..... map $P \mapsto h(P)$ from A_τ to $\mathbb{C}$ ⏲ $\sim 10\%$

$$H_K(1) \subseteq \mathrm{CM}_K(2) = K(\lambda_k(\tau) \,:\, k \in \{1,2,3\})$$

where $\lambda_k(\tau)$ are in terms of theta constants (analytic functions).

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\mathbf{H} & = & \mathrm{Aut}(\mathrm{CM}_K(2)/H_K(1)) & = & & & \langle\sigma_2\rangle_2 \\
\mathbf{G}/\mathbf{H} & = & \mathrm{Aut}(H_K(1)/K) & = & \langle\sigma_1\rangle_8 & &
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$$\tilde{p}_1(X) = \prod_{\sigma \in \mathbf{G}/\mathbf{H}} \left(X - \left(\tilde{\lambda}_1(\tau) + \tilde{\lambda}_1(\tau)^{\sigma_2}\right)^{\sigma}\right).$$

$$\approx X^8 - 36.591404X^7 + 530.70821X^6 - 3769.6792X^5 + \cdots$$

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Use an algorithm involving LLL to find

$$dp_1(X) = dX^8 + (9415170754271100000y^2 + 186957159620069475000)X^7 +$$
$$(3825208465541329538000y^2 + 8749908322905010620800000)X^6 + \ldots$$

where $d = 5^8\,7^4\,13^4\,23^4$.

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$p_1(X)$ is a degree 8 irreducible polynomial in $K[X]$. So

$$H_K(1) = K(\alpha) \qquad \text{where } p_1(\alpha) = 0.$$

where $K = \mathbb{Q}(y)$ where $y^4 + 24y^2 + 28 = 0$.